

The modelling of a Josephson junction and Heun polynomials

S.I. Tertychniy

VNIIFTRI, Russia

Abstract. The first order nonlinear ODE $\dot{\varphi}(t) + \sin \varphi(t) = q(t)$, $q(t) = B + A \cos \omega t$, where A, B, ω are real constants, is considered, the transformation converting it to a second order linear homogeneous ODE with polynomial coefficients is found. The latter is identified as a particular case of the double confluent Heun equation. The series of algebraic constraints on the constant parameters is found whose fulfillment leads to the existence of solutions representable through polynomials in explicit form. These polynomials are found to constitute the orthogonal normalizable system.

Nowadays, electronic devices based on the Josephson effect in superconductors and, in particular, Josephson junctions (JJ) [1] play the important role in the measurement technique, serving, in particular, the core element of the modern voltage standards [2]. The application needs lead to the growing importance of the theoretical and mathematical modelling of JJ properties. One of the theoretical tools commonly used for this purpose is the RSJ (Resistively Shunted Junction) model [3, 4] which applies, in the case of *overdamped* JJ [5], the ODE

$$\dot{\varphi}(t) + \sin \varphi(t) = q(t). \quad (1)$$

Here the (real valued) function $\varphi(t)$ called *the phase* and describing JJ state is unknown while $q(t)$ representing the external impact to JJ (the appropriately normalized *bias current* supplied by an external source) is assumed to be given. The dot denotes the derivative with respect to the variable t (the appropriately normalized current time).

The goal of the present notes is the discussion of some results concerning the equation (1) and its solutions in the particular case of harmonic $q(t)$ most important for applications. Thus we assume, without loss of generality,

$$q(t) = B + A \cos \omega t, \quad (2)$$

where A, B, ω are some constants subject to the condition $A \neq 0 \neq \omega$ in order to eliminate trivial situations.

As it was first noted by V.M. Buchstaber (see [6]), the first order nonlinear ODE (1) is equivalent, for arbitrary $q(t)$, to the system of two linear ODEs

$$\begin{aligned} 2\dot{x}(t) &= x(t) + q(t)y(t), \\ 2\dot{y}(t) &= -[q(t)x(t) + y(t)], \end{aligned} \quad (3)$$

where $x(t), y(t)$ are the new unknowns. They are connected with $\varphi(t)$ by the equation

$$\exp(i\varphi(t)) = \frac{x(t) - iy(t)}{x(t) + iy(t)}. \quad (4)$$

Thus φ is twice the phase of the complex quantity $x - iy$.

In the case of $q(t)$ defined by Eq. (2), the replacing of the independent real variable t with the complex valued variable

$$z = \exp(i\omega t), \quad (5)$$

translates Eqs. (3) to the equations

$$\begin{aligned} 4i\omega z \tilde{x}(z)' &= 2\tilde{x}(z) + [2B + A(z + z^{-1})] \tilde{y}(z), \\ 4i\omega z \tilde{y}(z)' &= -[2B + A(z + z^{-1})] \tilde{x}(z) - 2\tilde{y}(z). \end{aligned} \quad (6)$$

Here the separate notations for the unknowns x, y considered as functions of the complex variable z , $\tilde{x}(z) = x(t)$, $\tilde{y}(z) = y(t)$, are employed, the prime denotes the derivative with respect to z . Multiplying them by z , the equations with polynomial coefficients result.

One may extend the original meaning of \tilde{x}, \tilde{y} , treating them as analytic functions of the complex variable z which satisfy Eq. (6) everywhere in the complex plane except singular points. Then, obviously, a non-zero real or imaginary parts of any pair of such functions which prove smooth on some segment of the unit circle in \mathbb{C} yield a *real* solution of Eqs. (3) and, consequently, a (real) solution of Eqs. (1),(2).

Now let us consider the following transformation replacing the unknowns \tilde{x}, \tilde{y} with the functions $v = v(z), \tilde{v} = \tilde{v}(z)$ in accordance with equations

$$\begin{aligned} v &= i z^{-\frac{B}{2\omega}} \exp\left(\frac{A}{4\omega}(-z + z^{-1})\right) (\tilde{x} - i\tilde{y}), \\ \tilde{v} &= (2\omega z)^{-1} z^{-\frac{B}{2\omega}} \exp\left(\frac{A}{4\omega}(-z + z^{-1})\right) (\tilde{x} + i\tilde{y}). \end{aligned} \quad (7)$$

It is easy to show that the fulfilment of Eqs. (6) is equivalent to the condition of the vanishing of the following two expressions

$$v' - \tilde{v} \text{ and } z^2 \tilde{v}' + \left(\frac{A}{2\omega}(z^2 + 1) + \left(\frac{B}{\omega} + 1\right)z\right) \tilde{v} + \frac{1}{4\omega^2}v \quad (8)$$

that, in turn, is equivalent to the fulfilment of the second order linear ODE with polynomial coefficients

$$\left[z^2 \frac{d^2}{dz^2} + \left(\frac{A}{2\omega}(z^2 + 1) + \left(\frac{B}{\omega} + 1\right)z\right) \frac{d}{dz} + \frac{1}{4\omega^2}\right] v = 0. \quad (9)$$

The latter has the only two singular points, $z = 0$ and $z = \infty$.

The Möbius transformations

$$\zeta = \frac{z + \alpha}{z - \alpha}, \quad (10)$$

where ζ is the new independent complex variable and α is an arbitrary non-zero complex number, leads to representations of (9) more clearly revealing the symmetry in the roles of its singular points. In particular, in the case $\alpha = i$, one gets the equation

$$\left[(1 - \zeta^2)^2 \frac{d^2}{d\zeta^2} + 2 \left(\left(\frac{B}{\omega} - \zeta\right)(1 - \zeta^2) - 2i\frac{A}{\omega}\zeta\right) \frac{d}{d\zeta} + \frac{1}{\omega^2}\right] v = 0 \quad (11)$$

which proves to be is a particular instance of the *double confluent Heun equation* (DCHE) as it is given in [7], Eq. (4.5.11). It arises for the following set of parameters employed in [7]: $a = 0, c = -(B\omega^{-1} + 1), t = iA(2\omega)^{-1}, \tilde{\lambda} = (2i\omega A)^{-1}$. It is also worth noting that the canonical DCHE representation (Eq. (4.5.1) in [7])

$$z^2 \frac{d^2 y(z)}{dz^2} + (-z^2 + cz + t) \frac{dy(z)}{dz} + (-az + \lambda)y(z) = 0$$

results from (9) after the argument rescaling $z \rightarrow 2i\omega A^{-1}z$ and corresponds to the parameters $a = 0, c = B\omega^{-1} + 1, t = -(\frac{1}{2}A\omega^{-1})^2, \lambda = \frac{1}{4}o^{-2}$.

At the same time the most elegant form of the Möbius-transformed Eq. (9) results with $\alpha = 1$ in which case one gets

$$\left[(1 - \zeta^2) \frac{d}{d\zeta} (1 - \zeta^2) \frac{d}{d\zeta} + 2 \left(\frac{B}{\omega} (1 - \zeta^2) - \frac{A}{\omega} (1 + \zeta^2) \right) \frac{d}{d\zeta} + \frac{1}{\omega^2} \right] v = 0 \quad (12)$$

(see [8]). Here the singular points $\zeta = \pm 1$ are just the images of the only singular points $z = 0, \infty$ of Eq. (9). Hence there are no more singular points for Eq. (12) and, in particular, the point $\zeta = \infty$ is regular. It is also easy to show that the only effect caused by the transition to the reciprocal variable, $\zeta \rightarrow 1/\zeta$, is the inversing of the sign of the parameter A in (12). Another ζ transformation which obviously preserves the form of Eq. (12) is the reflection $\zeta \rightarrow -\zeta$. It leads to the combined inversion of parameter signs $A \rightarrow -A, B \rightarrow -B$. Thus, the signs of the both parameters A and B are irrelevant in the sense they can be (independently) reversed by means of transformations of the independent variable ζ .

Yet another useful representation of Eq. (9) results from the replacing of the unknown v by the new unknown $P \equiv P(z)$ by means of the substitution

$$v = \exp\left(-\frac{A}{2\omega}z\right) P. \quad (13)$$

It is convenient to represent the resulting equation as follows

$$z(zP' - nP)' - \mu z(zP' - nP) + (\mu - z)P' + \lambda P = 0, \quad (14)$$

where the following new *constant parameters* n, μ, λ replacing the equivalent triplet A, B, ω in accordance with definitions

$$n = -\left(\frac{B}{\omega} + 1\right), \quad \mu = \frac{A}{2\omega}, \quad \lambda = \frac{1 - A^2}{4\omega^2} = \frac{1}{(2\omega)^2} - \mu^2 \quad (15)$$

are utilized.

The representation (14) enables one to conjecture that if the parameter n assumes a non-negative integer value then this equation may admit *polynomial* solutions. Indeed, for $n = 0$, a constant is its solution for arbitrary ω , provided the constraint $\lambda = 0$ is additionally obeyed. Further, for a positive integer n , let us apply the ansatz

$$P = P_n = \sum_{k=0}^n a_k z^k, \quad (16)$$

where the constant coefficients a_k (dependent also on n) have to be determined. Substituting P_n into (14), one gets the following system of $n + 1$ linear homogeneous equations

$$0 = \lambda a_0 + \mu a_1, \quad (17)$$

$$0 = \mu(n - k + 1)a_{k-1} + (\lambda - k(n - k + 1))a_k + \mu(k + 1)a_{k+1} \quad \text{for } k = 1, \dots, n - 1, \quad (18)$$

$$0 = \mu a_{n-1} + (\lambda - n)a_n, \quad (19)$$

for $n = 1$ the subsystem (18) being void. It admits a nontrivial solution if and only if the determinant $\Delta_n(\lambda, \mu)$ of the following $(n + 1) \times (n + 1)$ dimensional 3-diagonal matrix

$$\Phi = \begin{pmatrix} \lambda & \mu \cdot 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \mu \cdot n & \lambda - 1 \cdot n & \mu \cdot 2 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & \mu \cdot (n - 1) & \lambda - 2 \cdot (n - 1) & \mu \cdot 3 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu \cdot (n - 2) & \lambda - 3 \cdot (n - 2) & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \lambda - (n - 3) \cdot 4 & \mu \cdot (n - 2) & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & \mu \cdot 3 & \lambda - (n - 2) \cdot 3 & \mu \cdot (n - 1) & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & \mu \cdot 2 & \lambda - (n - 1) \cdot 2 & \mu \cdot n \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & \mu \cdot 1 & \lambda - n \cdot 1 \end{pmatrix} \quad (20)$$

vanishes.

$$\Delta_n(\lambda, \mu) \equiv \det \Phi = 0 \quad (21)$$

is the algebraic equation, of degree $n + 1$ in λ , constraining the parameters μ, λ . Its fulfilment is the necessary and sufficient condition for the existing of polynomial solutions of Eq. (14).

If (21) is fulfilled, the polynomial coefficients a_k can be calculated as follows. Having introduced their appropriately scaled ratios R_k by means of the definition

$$R_k = \frac{\mu}{k} \frac{a_{k-1}}{a_k} \Rightarrow a_k = a_n \mu^{k-n} \frac{\Gamma(n+1)}{\Gamma(k+1)} \prod_{j=k+1}^n R_j \quad \text{for } k = 1, 2, \dots, n - 1, \quad (22)$$

Eqs. (18) are equivalent to the recurrence relations

$$R_k = 1 + \frac{\lambda}{k(k - n - 1)} + \frac{\mu^2}{k(k - n - 1)R_{k+1}}, \quad k = 1, 2, \dots \quad (23)$$

specifying, generally speaking, a continued fraction. However, for positive integer n , it is truncated at the $(n - 1)$ 'th step in view of Eq. (19) which is equivalent to the equation

$$R_n = 1 - \lambda/n. \quad (24)$$

Then, making use of Eqs. (23),(24), one may determine, step by step, all R_k . In turn, they determine coefficients a_k up to an arbitrary common factor.

Further, let us note that Eq. (22) can be represented in the following matrix form

$$\begin{bmatrix} R_k \\ 1 \end{bmatrix} = (Z_k R_{k+1})^{-1} \mathbf{M}_k \begin{bmatrix} R_{k+1} \\ 1 \end{bmatrix}, \quad (25)$$

$$\begin{aligned} \text{where } Z_k &= k(k - n - 1), \\ \mathbf{M}_k &= \begin{pmatrix} Z_k + \lambda & \mu^2 \\ Z_k & 0 \end{pmatrix}. \end{aligned} \quad (26)$$

Eq. (25) can also be interpreted as a linear map on the projective vector space of 2-element columns defined up to a nonzero factor. Iterating it and making use of (24), one gets

$$\begin{bmatrix} R_k \\ 1 \end{bmatrix} = \left[\prod_{j=k}^{n-1} Z_j \prod_{j=k+1}^n R_j \right]^{-1} \cdot \prod_{j=k}^{n-1} \mathbf{M}_j \times \begin{bmatrix} 1 - \frac{\lambda}{n} \\ 1 \end{bmatrix}. \quad (27)$$

Here the symbol \prod_{\rightarrow} denotes the product of matrices, where the factors corresponding to *larger* indices j are situated *at right* with respect to the lower index ones.

Eqs. (23)-(27) determine the set of coefficients a_k and, then, the polynomial $P_n(z)$ for *arbitrary* λ, μ irrespectively of the fulfillment of Eq. (21). However, if the latter is not satisfied, such a polynomial cannot be a solution of Eq. (14). Then some of Eqs. (17)-(19) have to be not satisfied as well. Since (18),(19) are automatically obeyed by the very meaning of Eqs. (23)-(27), it is the fulfillment of Eq. (14) which is equivalent to the equation

$$R_1 = -\frac{\mu^2}{\lambda}, \quad (28)$$

which is, in turn, connected with fulfillment of Eq. (17). Accordingly, if R_1 is considered as the result of calculation with the help of the formula (27) for $k = 1$, the equation

$$\begin{bmatrix} 1 \\ \frac{\mu^2}{\lambda} \end{bmatrix}^T \times \prod_{j=1}^{n-1} \mathbf{M}_j \times \begin{bmatrix} 1 - \frac{\lambda}{n} \\ 1 \end{bmatrix} = 0 \quad (29)$$

arises as the necessary condition of the fulfillment of Eqs. (17)-(19). In view of the close resemblance of algebraic structures of Eqs. (21) and (29), it is then natural to suppose that the left-hand-side expression of Eq. (29) is intimately connected with $\Delta_n(\lambda, \mu)$. It is, indeed, the case, and the following representation of $\Delta_n = \det \Phi$ through the product of a finite set of 2×2 matrices \mathbf{M}_j (26) takes place:

$$\Delta_n(\mu, \lambda) = - \begin{bmatrix} \lambda \\ \mu^2 \end{bmatrix}^T \times \prod_{j=1}^{n-1} \mathbf{M}_j \times \begin{bmatrix} n - \lambda \\ n \end{bmatrix}. \quad (30)$$

Similarly, there is the following explicit representation of the polynomial coefficients a_k through the analogous matrix products:

$$a_k = a_n \frac{(-\mu)^{k-n}}{k(n+1-k)!} \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \times \prod_{j=k}^{n-1} \mathbf{M}_j \times \begin{bmatrix} n - \lambda \\ n \end{bmatrix}, \quad k = 1, \dots, n-1. \quad (31)$$

Although, formally, it does not cover the case $k = 0$, a minor modification allows to compute a_0 as well (one has to replace *in the multipliers* the integer parameter k with $k + \varepsilon$, where ε is a small real number, to carry out computation with $k = 0$, and to pass to the limit $\varepsilon \rightarrow 0$ in the result).

The polynomials P_n reveal a remarkable symmetry concerning the coefficients in front of the “small” and “large” z powers. It can be discovered considering the following polynomial constructed from P_n :

$$\tilde{P}_n(z) = z^n [P'_n(z^{-1}) - \mu P_n(z^{-1})]. \quad (32)$$

A straightforward computation shows that it obeys Eq. (14) if and only if $P_n(z)$ does. However, the polynomial solution of Eq. (14) is unique up to a normalization. Indeed, the second solution linearly independent with $P_n(z)$ admits the following representation in quadratures:

$$Q_n = P_n \int z^n \exp(\mu(z + z^{-1})) P_n^{-2} dz \quad (33)$$

(*the associated functions*). As opposed to P_n , it is obviously singular in the point $z = 0$. Thus, having expanded \tilde{P}_n through the basis P_n, Q_n , it cannot involve any fraction of Q_n in its ‘content’ and, thus, $z^n(P'_n(z^{-1}) - \mu P_n(z^{-1})) \propto P_n(z)$. The constant proportionality coefficient can be fixed evaluating the equations above at the point $z = 1$. In this way one may obtain

$$P'_n(z) - \mu P_n(z) = \epsilon(2\omega)^{-1} z^n P_n(z^{-1}), \quad (34)$$

where $\epsilon^2 = 1$.

It is therefore shown that the fulfilment of Eq. (34) is the necessary condition for the polynomial $P_n(z)$ of degree n to satisfy Eq. (14). (In the general case, the formula (32) yields *the second*, linearly independent solution of (14)). Conversely, it is easy to prove that if *any analytic function* obeys Eq. (34) then Eq. (14) is also satisfied.

It is also worth noting the following representation of the phase function through the polynomial P_n :

$$\exp(-i\varphi(t)) = i\epsilon z^{n+1} \frac{P_n(z^{-1})}{P_n(z)}. \quad (35)$$

It follows from Eqs. (4),(7),(13),(34).

Eq. (34) leads to the following relations mentioned above among the coefficients of P_n :

$$\epsilon(2\omega)^{-1} a_0 = -\mu a_n \quad (36)$$

$$\epsilon(2\omega)^{-1} a_k = (n+1-k)a_{n+1-k} - \mu a_{n-k}, \quad k = 1, 2 \dots n, \quad (37)$$

where Eq. (36) can be considered as a particular case of Eq. (37), provided one has introduced $a_{n+1} \equiv 0$. As the above speculation claims, they are equivalent to Eq. (14). This implies some further relationships discussed below.

The elements of the matrix $\mathbf{G}^{(\epsilon)}$ of the linear system (37),(36) can be represented in terms of the Kronecker delta symbols as follows:

$$G_{jk}^{(\epsilon)} = \epsilon(2\omega)^{-1} \delta_{jk} + \mu \delta_{jn-k} - j \delta_{jn+1-k}, \quad j, k = 0, 1, \dots, n. \quad (38)$$

Then the product of the (*commuting*) matrices $\mathbf{G}^{(+1)}$, $\mathbf{G}^{(-1)}$ is easily computable and one gets

$$(\mathbf{G}^{(+1)} \cdot \mathbf{G}^{(-1)})_{jl} = (\lambda - j(n+1-j))\delta_{jl} + (n-j)\mu\delta_{j,l-1} + j\mu\delta_{j,l+1}. \quad (39)$$

The right-hand-side expression here is nothing else but the component representation of the matrix Φ (20). Thus Φ is factorizable, provided the constraint $4\omega^2(\lambda + \mu^2) = 1$ (see Eqs. (15)) is taken into account. The same concerns the determinants which are factorized as follows:

$$\Delta_n(\lambda, \mu) = \det \mathbf{G}^{(+1)} \cdot \det \mathbf{G}^{(-1)}.$$

Therefore the spectral equation (20) is equivalent to the condition

$$\text{either } \det \mathbf{G}^{(+1)} = 0 \text{ or } \det \mathbf{G}^{(-1)} = 0, \quad (40)$$

where the either branch involves the algebraic equation restricting parameters μ, ω (and depending on n). This property is useful in numerical applications.

Finally, let us consider polynomial solutions of the *two* specimens of Eq. (14) with common parameter μ but different integer n 's and λ 's obeying Eq. (21) which we denote $n_1, n_2, \lambda^{(1)}, \lambda^{(2)}$. An automatic computation establishes the validity of the following identity

$$0 = \frac{d}{dz} \left(z^{-(n_1+n_2)/2} \exp(-\mu(z + z^{-1})) \times \left[P_{n_2} \frac{dP_{n_1}}{dz} - P_{n_1} \frac{dP_{n_2}}{dz} - \frac{1}{2}(n_1 - n_2)z^{-1}P_{n_1}P_{n_2} \right] \right) + \Xi_{n_1, n_2} P_{n_1} P_{n_2}, \quad (41)$$

$$\begin{aligned} \text{where } \Xi_{n_1, n_2} = & z^{-(n_1+n_2)/2} \exp(-\mu(z + z^{-1})) \times \\ & \left[(\lambda^{(1)} - \lambda^{(2)} - \frac{1}{4}(n_1 - n_2)(n_1 + n_2 + 2)) z^{-2} \right. \\ & \left. + \frac{1}{2}\mu(n_1 - n_2)z^{-1}(1 + z^{-2}) \right]. \end{aligned} \quad (42)$$

It implies the *theorem*:

For $\mu > 0$ the polynomial solutions of Eq. (14) of different degrees n_1, n_2 are orthogonal on the semi-axis \mathbb{R}^+ with the weights Ξ_{n_1, n_2} (42), i.e.

$$\int_0^\infty \Xi_{n_1, n_2} P_{n_1} P_{n_2} dz = 0. \quad (43)$$

Under the same condition the polynomials P_n are also normalizable in appropriate norm involving the factor $\exp(-\mu(z + z^{-1}))$.

Resuming, the nonlinear first order ODE (1) arising in the theory of Josephson junctions with r.h.s. (2) was shown to be equivalent to the linear homogeneous second order ODE with polynomial coefficients (11). The latter equation was identified as a particular instance of the double confluent Heun equation. It means that *the problem (1),(2) proves completely solvable in terms of the double confluent Heun functions*. The series of constraints on the problem parameters enumerated by a non-negative integer parameter n (see (15)) was derived whose fulfilment leads to the existence of solutions representable in terms of polynomials (P_n) of degree n . The corresponding master equation is (14), the constraint equations are (21), where Δ_n can be computed by means of Eq. (30) through the finite products of 2×2 matrices with a single zero element, the polynomial coefficients are determined by Eq. (31) through the similar

matrix products. A curious first order linear “non-classical” two-argument differential equation (34) which P_n has to obey was found. The polynomial solutions of Eq. (14) were shown to constitute the normalizable orthogonal system on the positive semi-axes \mathbb{R}^+ .

References

- [1] Josephson B D 1962 *Phys. Lett.* **1**, 251; *Rev. Mod. Phys.* 1964 **36**, 216
- [2] Hamilton A H 2000 *Rev. Sci. Instrum.* **71**, 3611
- [3] McCumber D E 1968 *J. Appl. Phys.* **39**, 3113
- [4] Stewart W C 1968 *Appl. Phys Lett.* **12**, 277
- [5] Barone A and Paterno G 1982 *Physics and Applications of the Josephson Effect* (N.Y.: Willey)
- [6] Buchstaber V M, Karpov O V and Tertychniy S I 2002 *Abstracts of CPEM-2002* (Ottawa, Canada 2002), 502
- [7] Slavyanov S Yu and Lay W 2002 *Special functions: A Unified Theory Based on Singularities* (SPb: “Nevskiy dialect”), in Russian; there is also English edition, (Oxford, N.Y.: *Oxford Univ. Press*, 2000)
- [8] Tertychniy S I 2005 *Preprint math-ph/0512058*, p. 21